5.1 A change in perspective completes the picture

I've learned so many things from math that it would be unfair to pinpoint which was the most important lesson. It's like asking "which of these is your favorite child?" However, if I were hard-pressed to do so, I would say that math has taught me that **if I only understand one perspective, one viewpoint, of a particular problem, I have not understood the problem**. And this goes further: **If I have not understood the problem, how can I find a solution?**

There are numerous examples in mathematics where looking at a problem in a completely new way was what was needed to bring about a solution — and this change in perspective, incidentally, often gave rise to a whole new way of thinking. In fact, most breakthroughs in math are achieved by a change in perspective — it is incredibly rare that a real breakthrough is reached by sheer brute forcing the already familiar perspective to a conclusion. I can attest to that from my own experience as a researcher, where successfully solving a problem after having been gridlocked on it for weeks or months on end came about due to a shift in perspective.

"OK," you might think, "that seems reasonable. But Judy, what do you mean by a 'change of perspective' within the context of math?" That's a great question! After all, isn't math just *math*? Let's take a look at a famous historical problem where a change of perspective revolutionized our way of thinking.

The medieval city of Königsberg, Germany (now Kaliningrad, Russia), was known for its quite unique layout: through its center flows the Pregel (now Pregolya) River, which splits into two arms, dividing the town into four parts. Seven bridges connected these four parts to each other, allowing its inhabitants to move about Königsberg. Figure 5.1 depicts a historical map of the city. This intricate geography created by the Pregel River seemed to present a tantalizing fascination to the people: it is said that in the 1730s, the citizens of Königsberg amused themselves by trying to find a path through the city that would cross

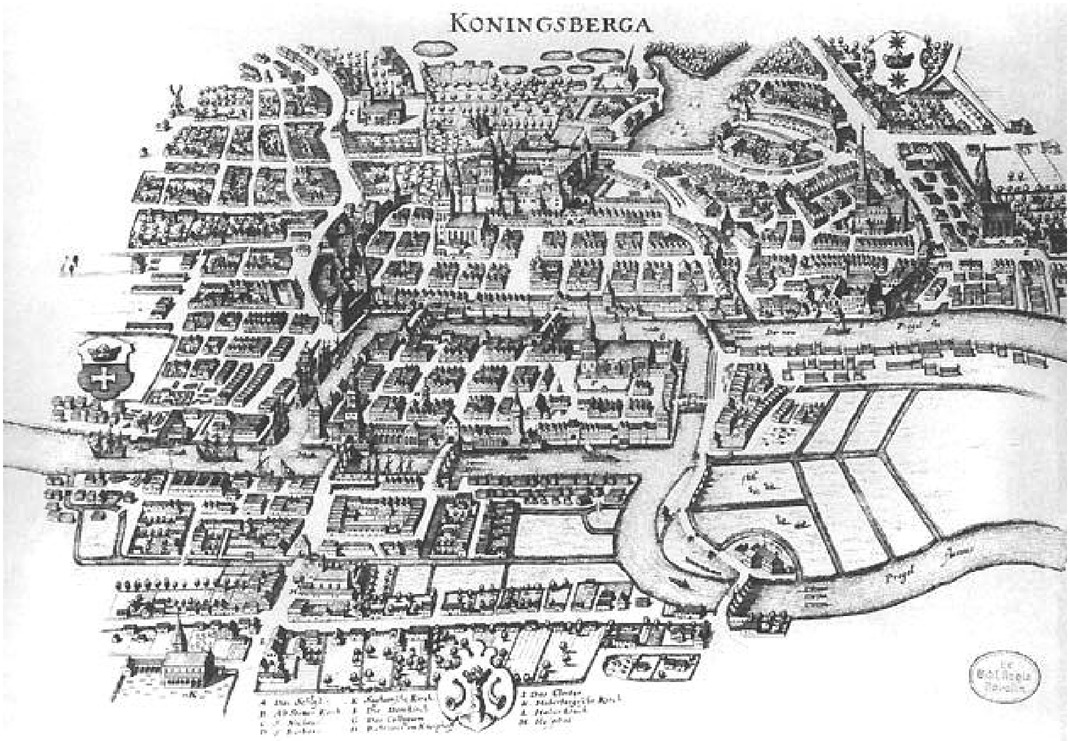
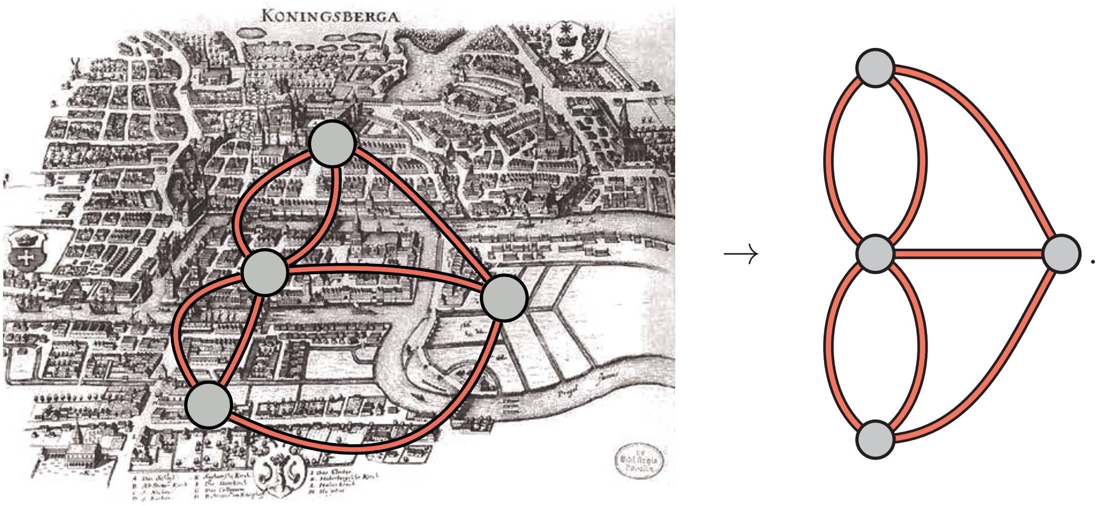


Figure 5.1: The historical center of Königsberg, Germany (now Kaliningrad, Russia), with its seven bridges across the various arms of the Pregel (now Pregolya) River.

each of the seven bridges exactly once. This proved to be a more challenging problem than originally expected, as the activity persisted for quite some years with no solution in sight. Try it out, dear reader, and see to which conclusion you come!

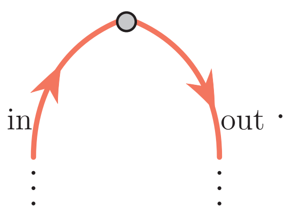
Having heard of this peculiar practice of the citizens of Königsberg, our very familiar friend, young Swiss mathematician Leonhard Euler, became intrigued — how were so many people trying to solve such a seemingly simple problem, yet to no avail? And so he set out to find a solution himself. However, instead of going through all paths one could possibly take through the city, Euler looked at the problem in a completely new way: to him, the city of Königsberg was essentially a collection of four land masses, that were connected (by bridges) in a particular way. So, to simplify things, he represented each mass by a single point, and connected the masses by lines if a bridge connected them:



(5.1)

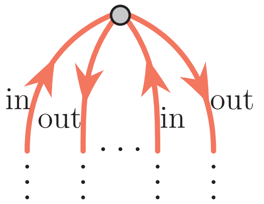
By focusing on the four points (land masses) and how they are connected to each other (by bridges), the actual geography of Königsberg becomes irrelevant. The problem boils down to finding a path through the resulting figure on the right-hand side of Eq. (5.1) that traces over each line exactly once (remember that each line represents a bridge and we want to cross each bridge exactly once and, in particular, not more than once). See what Euler did there? He took a step back from the situation, removed all the unimportant bells and whistles, and focused on the essence of the problem. And, by doing so, he found a solution:

Once Euler rephrased the original problem as finding a path through this conglomerate of points and lines, he had already almost solved the problem. It just required one more key insight: notice that if the path you choose takes you to a particular point via one line, it must take you away from that point via *another* line, since each line can only be used once. Let's call these the *in*-line and *out*-line, respectively,



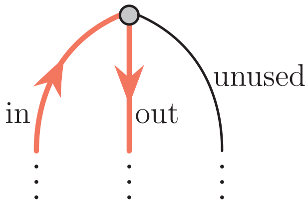
(5.2)

This seems obvious, I know, but it has an important consequence: lines connected to a point come in *pairs*, in-line and out-line. So if your path requires you to visit a particular point, maybe even multiple times, this point must have an *even* number of lines connected to it:



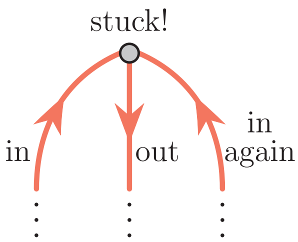
. (5.3)

Otherwise, you run the risk of not using a particular line,



, (5.4)

or getting stuck at the point altogether,



. (5.5)

But this means that, if a path through this network of points that uses each line exactly once is to be found, the points must be connected to an *even* number of lines! The *only* possible exceptions to this rule are the start and end points of the chosen path, as it doesn't matter whether you never return (to the start point) or get stuck (at the end point). But for all other points, the rule applies. So, *in order to find a path that traces each line exactly once, at most two points may be connected to an odd number of lines; all other points must be connected to an even number of lines.*

But if we now look back at the simplified drawing we devised of Königsberg in Eq. (5.1), we see that *all four* points are connected to an *odd* number of lines. So, this must mean that it is *impossible* to find a path through the city that crosses every bridge exactly once! What a bummer! But at least that explains why none of the inhabitants had managed to devise a path as per the requirements: it is simply not possible to do so.

This problem, now famously referred to as the *seven bridges of Königsberg problem*, may have started as a way for the Prussian elite to amuse themselves on Sunday afternoons, but its solution had far-reaching consequences: Euler's completely new way of viewing the problem as finding a path through a network of points inspired many mathematicians since to do the same and has spawned the field of *graph theory*, an area of math that is solely concerned with solving problems in terms of such networks. In fact, it is hard for me to overstate just how important graph theory is, not just in modern mathematics, but in modern *life*. The approach originally coined by Euler is the underlying mechanism of internet search engines, airline flights, and the infamous YouTube algorithm.

There is no such thing as understanding *completely*; there is only understanding *enough*.

But besides the far-reaching applications of graph theory, Euler's solution to the whimsical bridge problem is important on a personal level: **Opening one's mind to take on a completely new perspective broadens the horizon and not only opens up new possibilities that previously remained hidden but also provides a deeper, richer understanding of the situation at hand.** There is no such thing as understanding *completely*; there is only understanding *enough*. But who is to say what constitutes "enough?" If somebody brings a new viewpoint, don't dismiss it from the get-go. Give it real thought and reflection, mull it over, and try to understand it. For, even if this viewpoint is utterly nonsensical (which it rarely is), it still teaches you something, and there is no knowing where that can lead!